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Action correlation of orbits through non-conventional time reversal

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Abstract

Recently, Sieber and Richter calculated semiclassically a first off-diagonal contribution to the orthogonal form factor for a billiard on a surface of constant negative curvature. Following prior suggestions from the theory of disordered systems, they considered orbit pairs with almost the same action. For a generalization to systems invariant under non-conventional time reversal, which also belong to the orthogonal symmetry class, we show here that it is necessary to redefine the configuration space by a suitable canonical transformation; the distinction of this space is that it lets time reversal look conventional.

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1. Introduction

A wealth of numerical and experimental data on discrete (quasi-) energy spectra of quantum dynamics with global chaos in their classical counterparts reveals universality of spectral fluctuations on the scale of a mean-level spacing. The universal fluctuations are those predicted by the random-matrix theory [1, 2]. Such fidelity of chaotic dynamics to random-matrix theory was first conjectured by Bohigas *et al* [3]. A celebrated first step towards proving that conjecture was taken by Berry [4] on the basis of Gutzwiller's periodic-orbit theory [5]. The latter gives the density of levels as a sum over periodic orbits, each contributing a term of the structure $A_\gamma e^{iS_\gamma/\hbar}$, where S_γ is the action and A_γ is a stability coefficient independent of Planck's constant. The two-point correlator of the level density thus becomes a double sum with summands $\propto A_\gamma A_{\gamma'} e^{i(S_\gamma - S_{\gamma'})/\hbar}$. Berry argued that almost all contributions to that double sum must cancel one another since their action differences $S_\gamma - S_{\gamma'}$ are large compared to $\hbar$ and thus give rise to effectively uncorrelated phases $(S_\gamma - S_{\gamma'})\hbar^{-1}\mathrm{mod}(2\pi)$. The ensuing 'diagonal approximation' will be discussed below. Only very recently, an important step ahead was taken by Richter and Sieber [6] who discovered that, for systems with time reversal (TR) invariance, long periodic orbits with many self-crossings in the configuration space are accompanied by partner orbits of nearly the same action; an orbit and its partner differ only in that the partner narrowly avoids the crossing; everywhere else in configuration space both are very close.

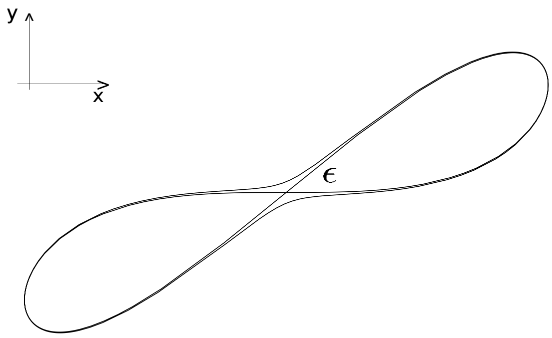

Figure 1. Sieber–Richter pairs in configuration space.

Correlations in the actions of periodic orbits and their relevance for spectral fluctuations were first discussed in [7]. Figure 1 schematically displays a Sieber–Richter pair of periodic orbits. Inasmuch as the action difference becomes smaller than Planck's constant in the limit of a small crossing angle, such pairs interfere constructively in the two-point correlator and its Fourier transform, the so-called form factor, and indeed turn out to give a correction to Berry's approximation in accordance with the random-matrix theory. Although Sieber and Richter worked out their ideas for chaotic billiards, it is expected that their result holds quite generally. Such an expectation is supported by previous analogous results from the theory of disordered systems [8].

Here we extend the work of Richter and Sieber to systems whose time reversal invariance is not conventional, with respect to which all momenta change sign while the coordinates remain unchanged. Let the conventional TR for a charged particle be broken by an external magnetic field, due to $H = \left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x})\right)^2 + V(\boldsymbol{x}) \longrightarrow THT^{-1} = \left(-\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x})\right)^2 + V(\boldsymbol{x}) \neq H$. But if the potential V, or in the case of billiards the boundary, is invariant under a spatial unitary symmetry U, such that $U^\dagger V U = V$ and $\left[U^\dagger\left(-\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x})\right)U\right]^2 = \left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{x})\right)^2$, the dynamics in question will enjoy invariance under the non-conventional time reversal (NCTR) $T_{\mathrm{n.c.}} = TU$, i.e. $T_{\mathrm{n.c.}} H T_{\mathrm{n.c.}}^{-1} = H$(for more explicit reasoning see below and [1]). While potentials with disorder cannot have symmetries U and thus cannot be time reversal invariant in the presence of a homogeneous magnetic field $\boldsymbol{B} = \mathrm{curl}\,\boldsymbol{A}$, non-conventional time reversal invariance for $\boldsymbol{B} \neq 0$ is common in atomic physics and can be designed for billards. Random-matrix theory predicts the same form factor for systems with conventional and non-conventional time reversal invariance. However, the Sieber–Richter interpretation of the form factor in terms of correlated orbits has to be modified for the NCTR-invariant Hamiltonians. It is sufficient to note that the action of an orbit in the magnetic field picks up an additional term equal to the magnetic flux through the orbit which is quite different for the members of the Sieber–Richter pair shown in figure 1 (one of the loops is traversed in the opposite direction); hence their actions cannot be approximately equal.

In the following, we will need the form factor $K^{\mathrm{GOE}}(\tau)$ for the Gaussian orthogonal ensemble of random matrices used for modelling chaotic TR and NCTR invariant Hamiltonians. The dimensionless argument $\tau = T/T_{\mathrm{H}}$ is the time T measured in units of the Heisenberg time $T_{\mathrm{H}} \propto 1/\hbar$. In the interval $0 < \tau \leqslant 1$ one has

$$K^{\mathrm{GOE}}(\tau) = 2\tau - \tau \log(1 + 2\tau). \tag{1}$$

As already mentioned above, that prediction is successful for time reversal invariant systems whose classical phase space is dominated by chaos.

A long-standing goal of periodic-orbit theory is to recover that prediction, at least in the sense of the Taylor series $K^{\mathrm{GOE}}(\tau) = 2\tau + \tau\sum_{k=1}^{\infty}(-2\tau)^k/k$ which converges for times up to half the Heisenberg time, $|\tau| \leqslant \frac{1}{2}$. The starting point is the periodic-orbit expression

$$K(\tau) = \lim_{\hbar\to 0}\frac{1}{T_{\mathrm{H}}}\left\langle\sum_{\gamma,\gamma'}A_\gamma A^*_{\gamma'}\mathrm{e}^{\mathrm{i}(S_\gamma - S_{\gamma'})/\hbar}\delta\left(T - \frac{T_\gamma + T_{\gamma'}}{2}\right)\right\rangle \tag{2}$$

where T_γ is the period of the orbit and the angular brackets demand average over a small time interval. Thus far, the first two terms in the Taylor series of $K^{\mathrm{GOE}}(\tau) = 2\tau - 2\tau^2\cdots$ have been established semiclassically; the first term, 2τ, represents Berry's diagonal approximation [4] and the second represents the contribution of near-action-degenerate pairs of orbits discovered by Sieber and Richter [6]. It is expected that the pairs of orbits differing by undergoing or avoiding crossings several times will eventually give the full expansion.

We propose to take another look at the Sieber–Richter pair of orbits in figure 1. One of its members contains a self-intersection with a small crossing angle ϵ and consists of two loops, one of which is passed clockwise and the other is passed counterclockwise. Playing with small deformations of such an orbit, it can be demonstrated that there exists a satellite orbit which is almost everywhere exponentially close to the original orbit; however, at one place in the configuration space the partner orbit has an avoided crossing at the place of a self-intersection [6]. The satellite can be regarded as nearly the same two-loop construction as the orbit with the crossing, except that it traverses both loops in the same sense. Time reversal symmetry is essential for the existence of the partner; otherwise retracing the motion along one of the loops would not be allowed. The difference of the actions of the orbit and its partner is

$$\Delta S = \frac{p^2\epsilon^2}{2m\lambda} + O(\epsilon^3) \tag{3}$$

where λ is the Lyapunov constant of the orbit. That action difference becomes of the order of $\hbar$ for sufficiently small angles ϵ. The breakthrough achieved by Sieber and Richter amounted to finding the number of self-crossings with an angle ϵ for orbits of a given long period T and to summing up the contribution of orbit pairs with small ϵ to the form factor in equation (2).

2. Non-conventional time reversal

We now propose to explain how the considerations of Sieber and Richter must be modified when invariance under conventional time reversal is replaced with non-conventional time reversal (NCTR) symmetry. To appreciate the necessity of modification it is worth stressing once more that even a very weak magnetic field destroys the near-degeneracy of the actions in each Sieber–Richter pair of orbits because of the totally different magnetic flux (due to the fact that one of the loops is passed in the opposite sense by the members of the pair). In a stronger field whose deforming impact on trajectories cannot be neglected, the pair depicted in figure 1 simply cannot exist because traversing a loop in two opposite senses is not allowed by the equations of motion.

The clue to our treatment lies in the fact that each periodic orbit of an NCTR-symmetric system has a twin with the same action. However, unlike for dynamics with conventional TR invariance, the twin is not the same orbit traversed backwards in time.

Now consider two-dimensional motion in a plane perpendicular to a uniform magnetic field $\boldsymbol{B} = B\boldsymbol{e}_z$ and suppose that the potential energy has the symmetry

$$V(x, -y) = V(x, y). \tag{4}$$

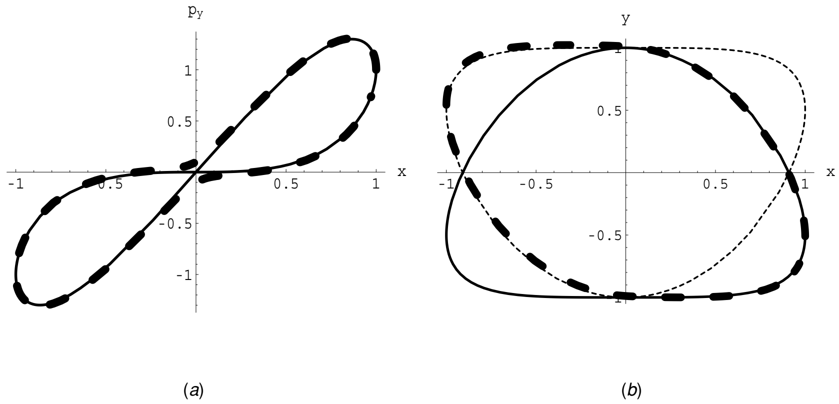

Figure 2. Sieber–Richter pairs in a NCTR system. Full line: the original orbit. Thin dashed line: NCTR twin of the orbit. Thick dashed line: partner orbit obtained by reconnecting parts of the orbit and its twin.

We will use the gauge $A_x = -By$, $A_y = A_z = 0$. Then the classical Hamiltonian

$$H = \frac{1}{2m}\left(p_x + \frac{eBy}{c}\right)^2 + \frac{p_y^2}{2m} + V(x, y) \tag{5}$$

will be invariant with respect to NCTR consisting of the conventional time reversal TR (changing the sign of the canonical momenta) followed by the reflection in the x-axis of the plane (replacement $y \to -y$, $p_y \to -p_y$). An arbitrary periodic orbit of our system with the trajectory

$$x = x(t) \qquad y = y(t) \tag{6}$$

and the momentum $p_y(t) = m\dot{y}(t)$ along the y-axis comes with the NCTR twin

$$\tilde{x}(t) = x(-t) \qquad \tilde{y}(t) = -y(-t) \qquad \tilde{p}_x(t) = -p_x(-t) \qquad \tilde{p}_y(t) = p_y(-t). \tag{7}$$

The twin has the same magnetic flux and action. Its trajectory is obtained from the original one by reflection in the x axis while the sense of traversal on both orbits is the same. Let us canonically transform the phase space variables as

$$x' = x \qquad y' = p_y \qquad p'_x = p_x \qquad p'_y = -y \tag{8}$$

and consider how our two orbits project onto the new configuration space $x'y'$. The original orbit will be described by the equations

$$x'(t) = x(t) \qquad y'(t) = p_y(t) \qquad p'_x(t) = p_x(t) \qquad p'_y(t) = -y(t) \tag{9}$$

while its NCTR twin obeys

$$\tilde{x}'(t) = x(-t) \qquad \tilde{y}'(t) = y'(-t) \qquad \tilde{p}'_x(t) = -p_x(-t) \qquad \tilde{p}'_y(t) = -p_y(-t). \tag{10}$$

After our canonical transformation (8), the NCTR under consideration appears in the guise of conventional time reversal. In particular, in the $x'y'$ alias xp_y plane the orbit twins are depicted by the same closed curve traversed in opposite directions.

Now it is easy to see that in the case of NCTR the Sieber–Richter arguments for the existence of the two-loop pairs of orbits of figure 1 remain fully valid. However, these pairs only exist in the xp_y plane (figure 2(*a*)) which is the configuration plane in the new coordinates. Only in that latter projection of the phase space to a two-dimensional submanifold switching the sense of traversal of a loop, as caused by the replacement of a crossing by an avoided crossing, is compatible with the equations of motion. Therefore, only in the xp_y plane do pairs as depicted in figure 1 exist and have nearly degenerate actions in strong magnetic fields. With that result in mind we may see that evaluating the number of Sieber–Richter pairs for a system with NCTR and calculating the form factor involves no new difficulties, relative to the case of conventional time reversal.

It is instructive to check what the pairs in figure 2(*a*) look like when the respective motion is projected onto the usual configuration space xy (figure 2(*b*)). They have little in common with the double-loop Sieber–Richter˙ pairs of figure 1. However, the general idea of building a new orbit with practically the same action by gently reconnecting parts of the original orbit and its NCTR twin is still valid. Depending on the projection chosen, the criterion for finding Sieber–Richter pairs changes. In systems with NCTR one may either look for the two-loop orbits with small opening angle in the xp_y plane or search for orbits such as in figure 2(*b*) in the xy plane. The latter may be preferable for systems like billiards in the magnetic field whose trajectory in the xp_y plane is discontinuous.

It may be somewhat puzzling that a jump in representation is needed for the Sieber–Richter treatment when the magnetic field is switched on; instead of xy space we must shift to xp_y. However, such a jump is only natural in view of the change of the universality class of the dynamics. A chaotic system with NCTR symmetry belongs to the Gaussian orthogonal ensemble only in the presence of the magnetic field. When the field is switched off the spectrum splits into two independent subspectra (even and odd with respect to $y \to -y$). The superposition of two such spectra whose levels may cross each other creates a specific ensemble obviously different from the GOE, usually called GOE × GOE.

Systems with a plane of symmetry in the uniform magnetic field constitute the most important but not the only conceivable example of NCTR. Consider, e.g., a two-dimensional system with a centre of symmetry

$$V(-x, -y) = V(x, y) \tag{11}$$

in an extremely non-uniform magnetic field

$$\boldsymbol{B} = y\boldsymbol{e}_z. \tag{12}$$

In the gauge $A_x = -By^2/2, A_y = A_z = 0$, the Hamiltonian of the system is invariant with respect to the NCTR composed of the time and spatial inversion which means that the momenta are unchanged. The partner obtained from a periodic orbit by this symmetry operation coincides with the original orbit if we draw it in the plane of momenta $p_x p_y$; hence it is in that latter plane that the Sieber–Richter pairs are described by a small intersection angle ϵ.

3. Establishing Sieber–Richter pairs in the surfaces of the section

In establishing the existence of a satellite orbit avoiding a given self-intersection Sieber and Richter linearized the dynamics around the crossing considered. The actual calculations make use of the fact that velocities and momenta are practically identical when conventional coordinates are employed. After our canonical transform the connection between the new ‘velocities’ and the momenta becomes more complicated. In particular, the direction of the momentum is no longer tangent to the trajectory in the new configuration space. This means that we must ascertain the existence of satellite orbits anew.

We start from a periodic orbit with a self-crossing under a small angle like the one shown in figures 1 and 2(*a*). A coordinate frame is introduced with its origin at the point of crossing and the x'-axis along the bisector of the small angle. Consider the Poincaré section at $x' = 0$

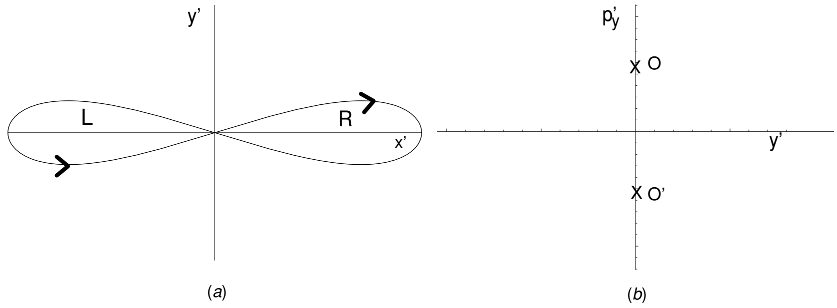

Figure 3. Poincaré map of the Sieber–Richter orbit (O) and its TR (NCTR) twin (O').

with the coordinates y' and p'_y on the surface of the section. The true Poincaré map (figure 3) is obtained when passages of the $x' = 0$ plane with a certain sign of $\dot{x}'$, say, $\dot{x}' > 0$, are marked. The self-crossing orbit will then be depicted by a periodic point O on the p'_y-axis with p'_y positive and small. The self-crossing TR (NCTR) twin of this orbit will produce another periodic point O' symmetrical with respect to the y'-axis: $y' = 0$, $p'_y < 0$.

We will concentrate, however, on the 'submaps' R and L of the Poincaré map describing the transform of y', p'_y generated by the right and left loops of the orbit. We will mark the crossing of the plane $x' = 0$ both for $\dot{x}' > 0$ and $\dot{x}' < 0$; such a breaking of Poincaré's rules is needed since R and L are not true Poincaré maps. We will also be interested in the TR (NCTR) submaps obtained by traversing the loops of the orbit in the direction opposite to the one shown in figure 3(a); these will be denoted as R' and L' respectively. The periodic point O of the total Poincaré map is simultaneously the periodic point of the submaps R and L whereas O' is the periodic point of the submaps L' and R'.

Each submap can be characterized by its stability matrix connecting the initial and final deflections of the y' coordinate and momentum from the periodic point of the respective submap. We will need M_R (the stability matrix of the right loop passed as it is shown in figure 3(a)) and $M_{L'}$ (the one for the left loop followed in the direction opposite to figure 3(a)). The large eigenvalues Λ_R and $\Lambda_{L'}$ of these two matrices can be evaluated as $\sim\exp(\lambda T)$ where λ is the Lyapunov constant and $T = T_R, T_{L'}$ is the period of the respective loop. The periods of the orbits in the sum for the form factor (2) are of the order $\hbar^{-1} \to \infty$; therefore, the larger eigenvalues of the stability matrices are exponentially large whereas the smaller ones $(1/\Lambda)$ are exponentially close to zero. The respective eigenvectors determine the unstable and stable directions of each of the submaps.

Consider figure 4 where periodic points O, O' of the submaps R, L' and their stable and unstable directions are shown. Let us investigate the application of the submap R to an initial point P chosen in the vicinity of the crossing A of the stable direction of R and unstable direction of L'. We shall represent the initial radius-vector by an expansion in powers of Λ_R^{-1},

$$\boldsymbol{r}_P = \boldsymbol{e}_s^R\left(l_{OA} + \frac{c_1}{\Lambda_R} + \cdots\right) + \boldsymbol{e}_u^R\left(\frac{l_{OB}}{\Lambda_R} + \frac{d_2}{\Lambda_R^2} + \cdots\right). \tag{13}$$

Here $\boldsymbol{e}_s^R, \boldsymbol{e}_u^R$ are the eigenvectors of the stability matrix of R along the stable and unstable directions, l_{OA} and l_{OB} are the distances from A and B to the periodic point O (B is the crossing of the unstable direction of R and the stable direction of L'); the coefficients $c_1, d_2, \ldots$ are so far undetermined.

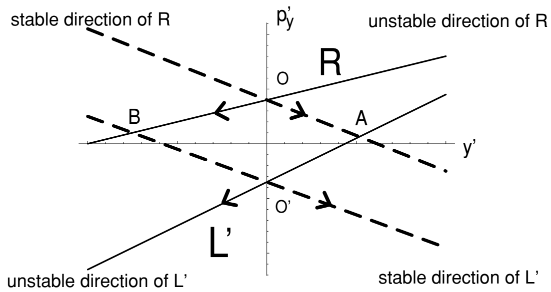

Figure 4. Stable and unstable manifolds of the right submap R and the time-reversed submap L'. A point P in the vicinity of A will be mapped to a point Q (not shown) in the vicinity of point B. Fine-tuning of the initial point P leads to a periodic orbit with avoided self-crossing.

After the loop R has been completed the point P will be mapped to the point Q obtained by squeezing along the stable and stretching along the unstable direction with the coefficient Λ_R:

$$\boldsymbol{r}_Q = \boldsymbol{e}_s^R\left(\frac{l_{OA}}{\Lambda_R} + \frac{c_1}{\Lambda_R^2} + \cdots\right) + \boldsymbol{e}_u^R\left(l_{OB} + \frac{d_2}{\Lambda_R} + \cdots\right). \tag{14}$$

It is seen that Q is infinitely close to the crossing point B of the unstable direction of R and the stable direction of L'; thus the distance of Q from the stable manifold of L' is exponentially small and depends on the coefficients $c_1, d_2, \ldots$.

Now consider the loop L' taking Q as its initial point. The loop will practically annihilate the stable component $\boldsymbol{e}_u^R\, l_{OB}$ and place the final point somewhere on its unstable manifold. The exact position of the final point on the unstable manifold of L' depends on $c_1,\ d_2, \ldots$; these can be fine-tuned so that the final point will coincide with the initial point P of the loop R[1]. But this would mean that P is a periodic point of the composition of the submaps R and L'. It corresponds to a new periodic orbit composed of the deformed loops R and L'. The new orbit crosses the 'true' Poincaré map at a point infinitely close to the crossing of the stable manifold of R and unstable manifold of L'. The TR (NCTR) twin of the new periodic orbit can be found by considering the sequence of submaps R' and L. The new orbit and its twin are of course Sieber–Richter partners with avoided crossing of the original orbit with self-intersection.

We have not used the connection between the velocities and momenta of the usual configuration space. Note also that the effect of the pair formation seems to remain in force even when the nonlinear corrections to the submaps R, L' are taken into account, with their stable and unstable manifolds depicted by curves rather than straight lines, as long as the stable and unstable manifolds intersect only once. For small angle ϵ, this condition should be fulfilled.

4. Conclusion

To summarize, the Sieber–Richter double-loop pairs of figure 1 or figure 2(a) may be observed only in that configuration space whose generalized coordinates are unchanged by the particular NCTR in reign. A periodic orbit and its NCTR twin are depicted in that space by the

[1] More accurately, demanding that the final point of the second loop coincides with the initial point P of the first loop, we obtain a set of equations for consecutive definition of $c_1, d_2, c_2, d_3, \ldots$. The coefficients in the expansion will not grow and convergence for $\boldsymbol{r}_P$ is guaranteed provided $\Lambda_R < \Lambda_L'$ which can be assumed wihout loss of generality.

same curve traversed in the opposite direction. This is the usual coordinate space xy in the systems with conventional time reversal symmetry; it is the xp_y plane for systems with NCTR $t \to -t,\ y \to -y$; finally, the $p_x p_y$ plane is so distinguished when NCTR is described by $t \to -t,\ x \to -x,\ y \to -y$. An attempt to break out of this symmetry-dictated space by a canonical transformation mixing the coordinates and momenta will immediately strip the Sieber–Richter pairs of their double-loop, intersection/avoided-crossing appearance.

However, using appropriate criteria the close action partners can be recognized in principle in any two-dimensional projection of the phase space.

Acknowledgment

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